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New Look at Gleason's Theorem for Signed Measures

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It is shown that the Gleason theorem holds not only for a finite but also for an $\mathfrak{n}$-finite signed measure m, where $\mathfrak{n}$ is a cardinal, defined on all closed subspaces of a Hilbert space whose dimension is a nonmeasurable cardinal $\neq 2$, if m is bounded from below on all one-dimensional subspaces.

1. INTRODUCTION AND PRELIMINARIES

In the quantum logic approach to the axiomatic foundation of quantum mechanics an important role is played by the quantum logic $\mathscr{L}(H)$ of all closed subspaces of a (not necessarily separable) Hilbert space H over the field C of real or complex numbers. A signed measure on $\mathscr{L}(H)$ is a function $m: \mathscr{L}(H) \to [-\infty, \infty]$ such that (1) $m(0)=0$; (2) m is σ-additive on all sequences of mutually orthogonal elements of $\mathscr{L}(H)$; (3) from the possible value $\pm\infty$ it attains at most one. A signed measure m is bounded if $\sup\{|m(M)|: M \subset H\} < \infty$. A positive signed measure is said to be a measure. The famous theorem of Gleason (1957) says that any finite measure m on a separable Hilbert space H, $\dim H \neq 2$, is in one-to-one correspondence with positive Hermitian operators T on H of finite trace via

$$m(M) = \operatorname{tr}(TM), \qquad M \in \mathscr{L}(H) \tag{1}$$

(we identify a subspace M with the orthoprojector P^M on it).

Sherstnev (1974) proved that formula (1) is also correct for all bounded, signed measures of a separable Hilbert space H, $\dim H \neq 2$. Drisch (1979) showed that the assumption of separability is superfluous when the Hilbert space is of nonmeasurable cardinality (for definition see below).

The situation with signed measures attaining infinite values is more complicated and it needs the following notions. By $\operatorname{Tr}(H)$ we denote the

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class of all bounded operators T in H such that, for every orthonormal basis $\{x_a: a \in A\}$ of H, the series $\sum_{a\in A}(Tx_a, x_a)$ converges and is independent of the basis used; the expression tr $T := \sum_{a\in A}(Tx_a, x_a)$ is called the trace of T.

A bilinear form is a function t: $D(t)\times D(t)\to C$ [$D(t)$ is a submanifold of H, not necessarily dense or closed in H, named the domain of definition of t], such that t is linear in both arguments, and $t(\alpha x, \beta y) = \alpha\bar{\beta}t(x, y)$, $x, y \in D(t)$, $\alpha, \beta \in C$. A bilinear form t is said to be symmetric if $t(x, y) = \overline{t(y, x)}$ for all $x, y \in D(t)$. A symmetric bilinear form t is called (1) positive if $t(x, x) \geq 0$ for all $x \in D(t)$; and (2) semibounded if there is a finite constant $K \geq 0$ such that $t(x, x) \geq -K$ for all $x \in D(t)$.

Let $P \in \mathscr{L}(H)$ and let $P \subset D(t)$. Then by $t\circ P$ we mean a symmetric bilinear form defined by $t\circ P(x, y) = t(Px, Py)$, $x, y \in H$. If $t\circ P$ is induced by a trace operator T, that is, $t\circ P(x, y) = (Tx, y)$, $x, y \in H$, then we say $t\circ P \in \mathrm{Tr}(H)$ and we define tr $t\circ P =$ tr T.

A signed measure m is said to be (1) f-bounded if $\sup\{|m(Q)|: Q \subset P\} < \infty$ whenever $|m(P)| < \infty$; (2) **n**-finite if there is a system of mutually orthogonal elements $\{M_a: a \in A\}$ such that $H = \oplus_{a\in A} M_a$ and $|m(M_a)| < \infty$ for each $a \in A$, and the cardinal of A is **n**. If $\mathbf{n} = \aleph_0$, we say that m is σ-additive. Here by $\oplus_{i\in I} P_i$ we mean the join of mutually orthogonal elements $P_i \in \mathscr{L}(H)$, $i \in I$. For any $0 \neq x \in H$ we denote by P_x the one-dimensional subspace of H spanned over x.

In Dvurečenskij (1985) it is proved that, for any σ-finite, f-bounded signed measure m on $\mathscr{L}(H)$ of a Hilbert space H whose dimension is a nonmeasurable cardinal $\neq 2$, $m(H) = \infty$, there is a unique symmetric bilinear form t with a dense domain $D(t)$ such that

$$m(P) = \begin{cases} \mathrm{tr}\ t\circ P & \text{if } m(P) < \infty \\ \infty & \text{elsewhere} \end{cases} \tag{2}$$

We recall, according to Ulam (1930), that the cardinal I is nonmeasurable if there is no nontrivial positive measure ν on the power set 2^A of a set A whose cardinal is I, such that $\nu(\{a\}) = 0$ for any $a \in A$. It is evident that all finite cardinals and $\aleph_0$ are nonmeasurable. Assuming the continuum hypothesis, **c** (cardinal of reals) is a nonmeasurable cardinal.

2. FRAME FUNCTIONS

The cornerstone of the Gleason theorem is the notion of a frame function. Denote $S(H) = \{x \in H: \|x\| = 1\}$. A function f: $S(H) \to [-\infty, \infty]$ is a frame function if (1) $f(\lambda x) = f(x)$ for all scalars $\lambda \in C$ with $|\lambda| = 1$; (2) there is a constant W (may be $\pm\infty$), named the weight of f, such that, for any orthonormal basis $\{x_a: a \in A\}$ of H, $\sum_{a\in A} f(x_a) = W$. A frame function

f has a finiteness property if $|\sum_{i\in I} f(x_i)| < \infty$, for some orthonormal system of vectors $\{x_i: i \in I\} \subset H$, implies $f|S(G)$ is a frame function with a finite weight, where $G = \oplus_{i\in I} P_{x_i}$. It is evident that any frame function with a finite weight has the finiteness property. A frame function f is regular if there is a positive symmetric bilinear form t with

$$D(t) = \{x \in H: x \neq 0, |f(x/\|x\|) < \infty\} \cup \{0\}$$

such that $f(x) = t(x, x)$ for any $x \in S(H) \cap D(t)$.

Let $\mathbf{n}$ be a cardinal. We say that a frame function f is $\mathbf{n}$-finite if there is a system of mutually orthogonal subspaces $\{M_a: a \in I\}$, $\oplus_{a\in I} M_a = H$, such that $f|S(M_a)$ is a frame function with finite weight for any $a \in I$, and the cardinal of I is $\mathbf{n}$. In particular, if $\mathbf{n} = \aleph_0$, we say that f is σ-finite. A frame function f is called (1) finite if $|f(x)| < \infty$ for any $x \in S(H)$; (2) bounded if $\sup\{|f(x)|: x \in S(H)\} < \infty$; and (3) semibounded if $\inf\{f(x): x \in S(H)\} > -\infty$.

Let $\mathbf{n}$ be a cardinal. We say that a function $m: \mathscr{L}(H) \to [-\infty, \infty]$ with $m(0) = 0$ and which, from the values $\pm\infty$, attains at most one is (1) $\mathbf{n}$-additive if, for any system of mutually orthogonal subspaces $\{M_a: a \in I\}$, card $I = \mathbf{n}$, we have

$$m\left(\bigoplus_{a\in I} M_a\right) = \sum_{a\in I} m(M_a) \tag{3}$$

and is (2) totally additive if equation (3) holds for any I with an arbitrary cardinal.

Proposition 1. Let m be a totally additive signed measure on a quantum logic $\mathscr{L}(H)$ of an arbitrary Hilbert space H. Then a map f defined via

$$f(x) = m(P_x), \qquad \|x\| = 1 \tag{4}$$

is a frame function with the finiteness property. Conversely, let f be a frame function with the finiteness property; then a map m on $\mathscr{L}(H)$ defined via

$$m(M) = \begin{array}{ll} 0 & \text{if } M = 0 \\ \sum_i f(x_i), & \{x_i\} \text{ is an orthonormal basis in } M \end{array} \tag{5}$$

is a totally additive function. This m is unique in the sense that (4) holds.

Proof. The first part of the proposition is evident. For the second part, we take into account that if the weight W of the frame function f is, for example, $+\infty$, then, for any orthonormal basis $\{x_a, a \in A\}$ in H and for any $\emptyset \neq A_1 \subset A$, $\sum_{a\in A_1} f(x_a) > -\infty$. In fact, if, for at least one $a \in A_1$, $f(x_a) = \infty$, then $\sum_{a\in A_1} f(x_a) = +\infty$. Now suppose $f(x_a) \neq +\infty$ for any $a \in A_1$. Let $K > 0$ be given. The weight W implies that there exists a finite $B_0 \subset A$ such that, for any finite B, $B_0 \subset B \subset A$, $\sum_{a \in B} f(x_a) > K$. Hence, for any finite $J \subset A_1$, $\sum_{J \cup B_0} f(x_a) > K$. Thus

$$\sum_{J \cup (B_0 \cap A_1)} f(x_a) > K - K_1 \tag{6}$$

where $K_1 = \sum_{B_0 \cap A_2} f(x_a)$ and $A_2 = A - A_1$.

Consequently, (6) implies that $\sum_{a \in A_1} f(x_a) \neq -\infty$.

The finiteness property of f entails that m is well defined, and m is totally additive on $\mathscr{L}(H)$. ■

Proposition 2. Let $3 \leq \dim H = n < \infty$ and let f be a frame function on H with the finiteness property and with an infinite weight. If $|f(x_i)| < \infty$, for $i = 1, \ldots, n-1$, and $|f(z)| < \infty$, where $x_i \perp x_j$ whenever $i \neq j$, then $z \in \alpha_1 x_1 + \cdots + \alpha_{n-1} x_{n-1}$ for some scalars $\alpha_1, \ldots, \alpha_{n-1} \in C$.

Proof. The same as Corollary 2 in Dvurečenskij (1986). ■

Theorem 3. Let $4 \leq \dim H < \infty$ and let f be a semibounded frame function with the finiteness property and infinite weight. If there are three mutually orthogonal vectors x_1, x_2, x_3 such that $\sum_{i=1}^{3} |f(x_i)| < \infty$, then f is a regular frame function.

Proof. Due to Proposition 2, we see that if we put

$$M = \{x \in H \colon x \neq 0, |f(x/\|x\|)| < \infty\} \cup \{0\}$$

then $M \in \mathscr{L}(H)$ and $\dim M \geq 3$. Proposition 1 says that $f|S(M)$ determines a signed measure m_M on $\mathscr{L}(M)$ via (5). Let $x \in M$, $\|x\| = 1$. Then

$$f(x) = m_M(M) - \sum_{i=1}^{r} f(x_i) \tag{7}$$

where $x_1, \ldots, x_r$ are mutually orthonormal vectors from M and orthogonal to x. Hence, $|f(x)| \leq |m_M(M)| + rK$, where $r = \dim M - 1$ and $K = -\inf\{f(y) \colon y \in S(M)\}$.

We proved $f|S(M)$ is a bounded frame function. Using the familiar assertion on bounded finite frame functions on finite-dimensional Hilbert space, we see that $f|S(M)$ is a regular frame function. ■

Theorem 4. Let H be a real or complex Hilbert space of dimension $\neq 2$. Then any semibounded frame function with a finite weight is regular. Moreover, there is a unique $T \in \mathrm{Tr}(H)$ such that

$$f(x) = (Tx, x), \qquad x \in S(H) \tag{8}$$

Proof. Define a map F on H via

$$F(x) = \begin{cases} 0 & \text{for} \quad x = 0 \\ \|x\|^2 f(x/\|x\|) & \text{for} \quad x \neq 0 \end{cases} \tag{9}$$

Now we shall define a semibounded symmetric bilinear form t defined on the whole of H. Let M be any three-dimensional subspace of H. As in (7), we see that $f|S(M)$ is a bounded frame function. Due to Dvurečenskij (1978), there is a positive Hermitian operator $T_M \in \mathrm{Tr}(M)$ such that $F(x) = (T_M x, x)$ for any $x \in M$.

Let now $x, y \in H$. Define $t(x, y) = (T_Q x, y)$, where Q is some two-dimensional subspace of H containing x, y. Since any two-dimensional Q may be embedded into some three-dimensional subspace M, we see that t is a well-defined symmetric semibounded bilinear form in question. In fact, if $x, y \in Q_1 \cap Q_2$

$$(T_{Q_1}x, y) = t(x, y) = (T_{Q_2}x, y)$$

Now we claim to show that t is a bounded bilinear form. Proposition 1 entails that, for any $M \in \mathscr{L}(H)$, there is a unique finite signed measure m_M on $\mathscr{L}(M)$ determined by $f|S(M)$.

1. Let $\dim M = n \geq 2$. Then, for any $x \in S(M)$,

$$t(x, x) = m_M(M) - \sum_{i=1}^{n-1} t(x_i, x_i)$$

where $x_1, \ldots, x_{n-1}$ is an orthonormal basis in $M \cap P_x^{\perp}$. Hence $t|S(M)$ is bounded.

2. Let $M = M_1 \oplus M_2$, and let $t|S(M_1)$ and $t|S(M_2)$ be bounded symmetric bilinear forms. We assert that so is $t|S(M)$.

Indeed, let $x \in S(M)$. Then $x = x_1 + x_2$, where $x_i = M_i x$, $i = 1, 2$. Calculate

$$t(x, x) = t(x_1, x_1) + t(x_2, x_2) + 2 \operatorname{Re} t(x_1, x_2)$$

By assumption, $|t(x_i, x_i)| \leq K_i \|x_i\|^2$, $i = 1, 2$, where $K_i = \sup\{|t(x, x)| : x \in S(M_i)\}$, $i = 1, 2$.

A bilinear form $s(f, g) := t(f, g) + K(f, g), f, g \in H$, where $K = -\inf\{t(x, x): x \in S(H)\}$ is a positive symmetric bilinear form. The Schwarz inequality implies

$$\operatorname{Re} s(f, g) \leq |\operatorname{Re} s(f, g)| \leq \{[t(f, f) + K\|f\|^2][t(g, g) + K\|g\|^2]\}^{1/2}$$

and

$$\operatorname{Re} t(f, g) \leq \{[t(f, f) + K\|f\|^2][t(g, g) + K\|g\|^2]\}^{1/2} - K \operatorname{Re}(f, g)$$

Therefore

$$2|\operatorname{Re} t(x_1, x_2)| \leq 2[(K_1 + K)(K_2 + K)]^{1/2}$$

This proves that $t|S(M)$ is bounded.

3. Here we show that t is bounded on $S(H)$. If not, then there exists $e_1 \in S(H)$ with $t(e_1, e_1) \geq 1$. Applying part 2 to $M_1 = P_{e_1}^{\perp}$, we see that $t|S(M_1)$ must be unbounded. Therefore, there is $e_2 \perp e_1$, $\|e_2\| = 1$, with $t(e_2, e_2) \geq 1$. Continuing, according to induction, we find after n steps a vector e_{n+1}, $\|e_{n+1}\| = 1$, orthogonal to $e_1, \ldots, e_n$ ($e_1, \ldots, e_n$ are orthonormal vectors) with $t(e_{n+1}, e_{n+1}) \geq 1$. Define $M = \bigoplus_{n=1}^{\infty} P_{e_n}$, and let m_M be a finite signed measure on $\mathscr{L}(M)$ from Proposition 1. Then

$$m_M(M) = \sum_{n=1}^{\infty} m_M(P_{e_n}) = \sum_{n=1}^{\infty} t(e_n, e_n) = \infty$$

which contradicts the finiteness of m_M, and, therefore, t is a bounded symmetric bilinear form.

Hence, there exists a unique Hermitian operator T on H such that

$$f(x) = t(x, x) = (Tx, x), \qquad \|x\| = 1$$

Finally, the finiteness of the weight of f gives us $T \in \mathrm{Tr}(H)$. ■

Theorem 5. Let H be a real or complex Hilbert space of dimension $\neq 2$ and let $\mathbf{n}$ be any cardinal. Then any $\mathbf{n}$-finite semibounded frame function with the weight belonging to $(-\infty, \infty]$ and with the finiteness property is regular.

Proof. If the weight W of f is finite, the assertion follows from Theorem 5.

Now let $W = +\infty$. Define a map F on H via (9). Put $D(F) = \{x \in H: F(x) < \infty\}$. We claim to show that $D(F)$ is a dense submanifold in H. Let $x, y \in D(F)$. Due to the $\mathbf{n}$-finiteness of f, we have that there exist three orthonormal vectors x_1, x_2, and x_3 and

$$0 \neq z := \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \perp x, y$$

and $Px \neq 0 \neq Py$, where $P = \bigoplus_{i=1}^{3} P_{x_i}$. Proposition 2 implies that $f|S(M)$, where $M = P_z \vee P_x \vee P_y$ is a finite frame function; hence $F(x+y) < \infty$. The $\mathbf{n}$-finiteness of f gives the density of $D(F)$.

Now we shall define a semibounded symmetric bilinear form t defined on $D(F)$. As in the proof of Theorem 4, we can prove that, for any two-dimensional $Q \subset D(F)$, there is $T_Q \in \mathrm{Tr}(H)$ with $F(x) = (T_Q x, x)$, $x \in M$. Defining $t(x, y) = (T_Q x, y)$ for some two-dimensional Q containing x, y we prove the theorem. ■

Remark 1. It is known (see also Proposition 6) that if dim $H = 2$, then there are finite frame functions that are not regular. On the other hand, not any bilinear form with a dense domain determines a frame function. In fact, let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of a separable Hilbert space H. Suppose that $\{e_n\}_{n=1}^{\infty}$ is a part of a Hamel basis $\{g_t\colon t\in T\}$. Fix a unit vector $g_{t_0}\in\{g_t\colon t\in T\}-\{e_n\}_{n=1}^{\infty}$ and define a linear operator B in H via $B(\sum_{t\in T_0}\alpha_t g_t)=\alpha_{t_0}g_{t_0}$, where T_0 is the finite part of T containing t_0, and α_t are scalars. The positive symmetric bilinear form $t(x, y) := (Bx, B_y)$, $x, y\in H$, does not determine a frame function, since $t(e_n, e_n)=0$, $n\geq 1$, and $t(g_{t_0}, g_{t_0})=1$. This example is from Lugovaja and Sherstnev (1980).

3. FINITE SIGNED MEASURES

As has been noted, Sherstnev (1974) generalized the Gleason theorem to bounded signed measures, remarking that (1) is true even if $\sup\{|m(P_x)|\colon x\in S(H)\}<\infty$. Drisch (1979) formulated his result only for bounded signed measures. In this section we show that for the validity of (1) both of the above conditions may be weakened.

The positive and negative variations m^+ and m^- of the signed measure m are defined as follows:

$$m^+(M)=\sup\{m(N)\colon N\subset M\}$$

$$m^-(M)=-\inf\{m(N)\colon N\subset M\}$$

for any $M\in\mathscr{L}(H)$. The total variation of m is the map $|m| := m^+ + m^-$. Some properties of variations of m are:

1. $m^+, m^-, |m|$ are nonnegative.
2. $m^+ = -m^-$ and $m^- = -m^+$.
3. If $M\subset N$, then $m^+(M)\leq m^+(N)$ and $m^-(M)\leq m^-(N)$.
4. $m^+(\bigoplus_{n=1}^{\infty} M_n)\geq\sum_{n=1}^{\infty} m^+(M_n)$ and $m^-(\bigoplus_{n=1}^{\infty} M_n)\geq\sum_{n=1}^{\infty} m^-(M_n)$.
5. $m^+(0)=m^-(0)=0$.
6. $|m(M)|\leq|m|(M)$ for all $M\in\mathscr{L}(H)$.
7. If $m\colon\mathscr{L}(H)\to(-\infty,\infty]$, then $m+m^-=m^+$;
   if $m\colon\mathscr{L}(H)\to[-\infty,\infty)$, then $-m+m^+=m^-$.

In conventional measure theory it is known that any finite signed measure has finite positive and negative variations. This result is not valid in quantum logics, in general; see also Sherstnev (1974).

Proposition 6. For any integer $n\geq 2$, there is an unbounded, finite, signed measure m (in fact infinitely many) on a quantum logic $\mathscr{L}(H)$ of a Hilbert space H, dim $H = n$.

Proof. Let $\dim H = 2$. Choose a sequence of one-dimensional subspaces $\{M_n\}_{n=1}^{\infty}$ which contains no orthogonal pairs. Define a function m on $\mathscr{L}(H)$ via

$$m(M) = \begin{cases} 0 & \text{if } M = 0 \\ 1 & \text{if } M = H \\ n+1 & \text{if } M = M_n \\ -n & \text{if } M = M_n \end{cases}$$

and on other one-dimensional subspaces M, $M^{\perp}$ we choose $m(M) \in \{2, -1\}$ such that $m(M) + m(M^{\perp}) = 1$.

Then m is a well-defined, unbounded, finite signed measure.

Let $n \geq 3$. First we take into account the result of Hamel (1905) that there exists a discontinuous additive functional φ: $R \to R$, where R is the set of all reals. For that it is sufficient to find a subset $S \subset R$ such that every real number r can be uniquely represented as $r = \sum_{i=1}^{n} \beta_i s_i$, where $s_i \in S$ and β_i is rational. Using the Zorn lemma, we may show that this S exists and it contains at least two (in fact card $S = c$) numbers s_1 and s_2, where s_1 is an irrational. If we put $\varphi(\sum_{i=1}^{n} \beta_i a_i) = \beta_1$, then φ is the functional in question.

Let now $T \neq KI$, where K is a real constant and I is the identity operator in H, be a Hermitian operator in H. Define a finite frame function $f(x) = \varphi((Tx, x))$, $x \in S(H)$. We assert that it determines a finite unbounded signed measure on $\mathscr{L}(H)$. Suppose the converse. Then f is bounded. According to Theorem 4, there exists a Hermitian operator U in H with $F(x) = (Ux, x)$, $x \in H$, where F is defined via (9). Consequently, F is continuous.

On the other hand, the set $\{(Tx, x): x \in S(H)\}$ is a finite closed interval $[a, b]$ in R, where a and b are the minimal and maximal proper values of T; hence $a \neq b$. It is evident that there are two rationals α_1 and α_2 such that $\alpha_1 s_1, \alpha_2 s_2 \in [a, b]$, $\alpha_1 \neq 0 \neq \alpha_2$, and $\alpha_1 s_1$, $\alpha_2 s_2$ are simultaneously either positive or negative. Also we may find a sequence of positive rationals, $\{\beta_n\}$ say, such that $\beta_n \to \beta = \alpha_1 s_1 / \alpha_2 s_2$. Choose two vectors $x_1, x_2 \in S(H)$ such that $(Tx_i, x_i) = \alpha_i s_i$, $i = 1, 2$. Define $y_n = \beta_n^{1/2} x_2$, $y = \beta^{1/2} x_2$. Then $y_n \to y$ and $F(y_n) = 0 \not\to F(y) = \alpha_1 \neq 0$, which is a contradiction. ■

Now we give the following characterization of finite signed measures on $\mathscr{L}(H)$.

Theorem 7. Let m be a finite signed measure on $\mathscr{L}(H)$, where H has a nonmeasurable cardinal $\neq 2$. The following assertions are equivalent:

(i) m is bounded.

(ii) m is representable via (1).
(iii) m^+ is a finite measure.
(iv) m^- is finite.
(v) $|m|$ is finite.
(vi) $\sup\{|m(P_x)|: \|x\|=1\}<\infty$.
(vii) $\inf\{m(P_x): \|x\|=1\}>-\infty$.
(vii) $\inf\{m(P_x): \|x\|=1\}>-\infty$.
(viii) $\sup\{m(P_x): \|x\|=1\}<\infty$.
(ix) There are two measures m_1 and m_2 such that $m=m_1-m_2$.

Proof. The equivalence of (i) and (ii) follows from the result of Drisch (1979). It is clear that (i) implies (iii).

Due to the identity $m(M)=m(H)-m(M^\perp)$, $M\in\mathscr{L}(H)$, we see that (iii) and (iv) are equivalent, and they are also equivalent to (v). The proposition (6), from the part describing the properties of variations m^+ and m^-, proves the implication (v)$\rightarrow$(i).

Let (i) hold. Then (vi) is valid, and (vi) implies (vii). The proof of Theorem 4 entails the validity of the implication (vii)$\rightarrow$(vi). Applying the proof of Theorem 4 to a finite signed measure $-m$, we prove the equivalence of (vi) and (viii).

Suppose that (vi) hold. First we claim to show that m is totally additive [even without the validity of assertion (vi)]. Let $\{M_a: a\in I\}$ be a system of mutually orthogonal elements belonging to $\mathscr{L}(H)$ with the join M. Define a finite signed measure μ on the σ-algebra 2^I of all subsets of a set I via $\mu(\emptyset)=0$, $\mu(A)=m(\bigoplus_{a\in A}M_a)$, $A\subset I$. It is known (e.g., Halmos, 1953) that there exists a Jordan decomposition for μ, $\mu=\mu^+-\mu^-$, where μ^+ and μ^- are positive measures on 2^I. Due to Ulam (1930), there are two subsets of I, D^+ and D^-, with at most countably many elements such that $\mu^+(I-D^+)=0$, $\mu^-(I-D^-)=0$. Put $D=D^+\cup D^-$. Then $\mu(I-D)=\mu^+(I-D)-\mu^-(I-D)$ and $0\le\mu^\pm(I-D)\le\mu^\pm(I-D^\pm)=0$. Thus $\mu(I-D)=0$.

Calculate

$$m(M)=m\left(\bigoplus_{a\in I}M_a\right)=\mu(I)=\mu(I\cap D)+\mu(I-D)=\mu(I\cap D)$$
$$=\mu(D)=\sum_{a\in D}\mu(\{a\})=\sum_{a\in D}m(M_a)$$

It is clear that, for any $a\notin D$, $\mu(\{a\})=0$. Hence, $m(M)=\sum_{a\in I}m(M_a)$.

The total additivity of m implies, in particular, $T\in\mathrm{Tr}(H)$, where T satisfies $m(P_x)=(Tx,x)$, $x\in S(H)$, according to Theorem 4. Then

$$m(M)=\sum_{i\in A}m(P_{f_i})=\sum_{i\in A}(Tf_i,f_i)=\mathrm{tr}(TM)$$

where $\{f_i: i\in A\}$ is an orthonormal basis in M.

Finally, let m hold for the Gleason theorem, that is, the formula (1) is valid. Putting $T = T^+ - T^-$, where T^+ and T^- are the positive and negative parts of the Hermitian operator T, we see that for $m_1(M) := \operatorname{tr}(T^+M)$ and $m_2(M) := \operatorname{tr}(T^-M)$, $M \in \mathscr{L}(H)$, we can obtain (ix).

Conversely, (ix) entails (i) immediately. ■

Theorem 8. (A. M. Gleason). Let $\mathbf{n}$ be a cardinal and let m be an $\mathbf{n}$-finite semibounded signed measure with $m(H) = \infty$ on a quantum logic $\mathscr{L}(H)$ of a Hilbert space H whose dimension is a nonmeasurable cardinal $\neq 2$. Then there is a unique semibounded symmetric bilinear form t with a dense domain such that (2) holds.

Moreover, if $m(\bigoplus_{a\in A} M_0) < \infty$, then

$$m\left(\bigoplus_{a\in A} M_a\right) = \sum_{a\in A} m(M_a)$$

Proof. Due to Theorem 7, the semiboundedness of m implies that, for any $P \in \mathscr{L}(H)$ with $m(P) < \infty$ we have $\sup\{|m(Q)|: Q \subset P\} < \infty$, that is, m is f-bounded. A simple modification of the proof of Lemma 4.4, from the paper of Dvurečenskij (1985), gives us the formula (2). ■

Remark 3. (i) The assertion of Theorem 10 remains valid even in the case when m is an $\mathbf{n}$-finite, semibounded, $\mathbf{m}$-additive function on a quantum logic $\mathscr{L}(H)$ supposing the dimension of H is a nonmeasurable cardinal $\neq 2$, and $\mathbf{n}$ and $\mathbf{m}$ are two cardinals such that $\mathbf{n} \leq \mathbf{m}$, $\aleph_0 \leq \mathbf{m}$.

(ii) We have seen that the semiboundedness of m implies the f-boundedness of m. I do not know whether the converse implication is true.

(iii) For an $\mathbf{n}$-finite measure it is known (Dvurečenskij, 1986) that m is totally additive and $m(M) < \infty$ iff $t \circ M \in \operatorname{Tr}(H)$. For signed measures on nonseparable Hilbert space quantum logic a similar proposition is unknown (see also Proposition 1). If there are two measures m_1 and m_2 on $\mathscr{L}(H)$ such that $m = m_1 - m_2$, then $|m(M)| < \infty$ iff $t \circ M \in \operatorname{Tr}(H)$. For a separable Hilbert space this equivalence is true.

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